J. Fluid Mech. (1967), *vol.* 30, *part* 4, *pp.* 651–661
Printed in Great Britain

On the occurrence of cellular motion in Bénard convection

By E. PALM, T. ELLINGSEN AND B. GJEVIK

Department of Mechanics, University of Oslo

(Received 21 April 1967)

The interval of Rayleigh numbers in Bénard convection corresponding to cellular motion is determined in the case of free-free boundaries, rigid-free boundaries and rigid-rigid boundaries, taking into account the variation of the kinematic viscosity with temperature. Neglecting the effect of surface tension, it is shown that this interval is largest for the rigid-rigid case. The most important feature from the obtained formula (6.1) is, however, that the interval is extremely dependent on the depth of the fluid layer. To obtain a cellular pattern it is therefore necessary to have very small fluid depths. For example, with Silicone oil and a fluid depth of 7 mm, cellular motion will, according to the theory, be observed for Rayleigh numbers larger than the critical value and less than 1·07 times the critical value. For a fluid depth of 5 mm, however, the formula (6.1) gives that cellular motion will be observed for Rayleigh numbers up to 1·54 times the critical value.

1. Introduction

When in experiments on thermal convection the Rayleigh number is given a value larger than the critical one, a motion will be set up which often may have a very regular pattern. The latest experimental results available concerning the cellular motion in thermal convection are those due to Koschmieder (1966), who demonstrates that in the case of rigid-free boundaries, a regular cellular motion develops, the cells being nearly of hexagonal form. In the case of rigid-rigid boundaries, however, he finds that other patterns are preferred, which are highly influenced by the lateral boundaries.

From the theoretical side it has been shown in works by Palm (1960), Segel & Stuart (1962), Palm & Øiann (1964) and Segel (1965) (referred to as I–IV, respectively) that for values of the Rayleigh number slightly above the critical one, hexagonal cells constitute the only stable motion. For larger values of the Rayleigh number both hexagons and two-dimensional rolls are stable solutions, whereas for still higher values, rolls are the only stable modes. To derive this result, it was essential to take into account the variation of viscosity with temperature. However, for mathematical simplicity only the free-free case was considered. In order to be able to compare theory and experiments quantitatively we shall therefore in the present note take into account more realistic boundary conditions, i.e. we shall consider the rigid-rigid case and the rigid-free case.

Schlüter, Lortz & Busse (1965), extending the work by Malkus & Veronis (1958), prove, under very general conditions, that when the variation of material properties with temperature is neglected, two-dimensional rolls are the only stable solutions. We shall here in part apply the same techniques and notations as used by Schlüter *et al.* From the results of the papers I–IV we are, however, led to take into account only two of all the possible solutions of the first-order equations.

2. The basic equations and boundary conditions

It is assumed that the fluid layer is of infinite horizontal extent and is bounded by two horizontal boundaries. When the Boussinesq approximations are applied, the equations of motion and continuity may be written

$$\frac{\partial u_i}{\partial t}+u_k\frac{\partial u_i}{\partial x_k} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x_i}-\frac{\rho g}{\rho_0}\delta_{i3}+\frac{\partial(\nu u_{ik})}{\partial x_k}, \tag{2.1}$$

$$\frac{\partial u_i}{\partial x_i} = 0, \tag{2.2}$$

where we have used the summation convention, and i, k may be $1, 2, 3$. x_1, x_2 are horizontal co-ordinates, x_3 the vertical co-ordinate measured positive upwards, t the time, u_i the velocity, ρ the density, ρ_0 a standard density, p the pressure, g the acceleration of gravity and ν the kinematic viscosity defined as the ratio of the viscosity divided by ρ_0. δ_{ij} is the Kronecker delta, and u_{ik} is the deformation tensor given by

$$u_{ik} = \frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}. \tag{2.3}$$

Furthermore, the equation for conduction of heat is

$$\frac{\partial T}{\partial t}+u_k\frac{\partial T}{\partial x_k} = \kappa\nabla^2 T \tag{2.4}$$

and the equation of state may be written

$$\rho = \rho_0(1-\alpha(T-T_0)). \tag{2.5}$$

Here T denotes the temperature, κ the (constant) thermal diffusivity, α the coefficient of expansion and T_0 is a standard temperature. It will be assumed that ν is a linear function of temperature

$$\nu = \nu_0(1+\gamma(T-T_0)), \tag{2.6}$$

where γ is a constant.

The temperature may be written

$$T = T_0-\beta x_3+\theta, \tag{2.7}$$

where $\beta = \Delta T/h$ with ΔT denoting the difference in the temperature between the lower and upper boundary and h the depth of the layer. The pressure is divided in two parts

$$p = p_s+\hat{p}, \tag{2.8}$$

where p_s is defined by

$$\frac{1}{\rho_0}\frac{\partial p_s}{\partial x_3} = -g(1+\alpha\beta x_3), \tag{2.9}$$

i.e. p_s is the pressure in the case of pure heat conduction with a temperature $T = T_0 - \beta x_3$. Applying (2.2) and (2.6), the viscous term in (2.1) may be written

$$\frac{\partial(\nu u_{ik})}{\partial x_k} = \nu_0 \nabla^2 u_i - \nu_0 \gamma\beta \frac{\partial(x_3 u_{ik})}{\partial x_k} + \nu_0 \gamma \frac{\partial(\theta u_{ik})}{\partial x_k}. \tag{2.10}$$

Let h denote the depth of the fluid layer. To get a dimensionless form of the equations we set

$$x_i = hx_i', \quad u_i = \kappa u_i'/h, \quad t = h^2 t'/\kappa, \quad \theta = \kappa\nu_0\theta'/\alpha g h^3,$$
$$\hat{p} = \kappa^2 \rho_0 p'/h^2, \quad u_{ik} = \kappa u_{ik}'/h^2.$$

This yields, applying (2.9) and (2.10), and dropping the primes,

$$\frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} = -\frac{\partial p}{\partial x_i} + \mathscr{P}\theta\delta_{i3} + \mathscr{P}\nabla^2 u_i - \Gamma\mathscr{R}\frac{\partial(x_3 u_{ik})}{\partial x_k} + \Gamma\frac{\partial(\theta u_{ik})}{\partial x_k}, \tag{2.11}$$

$$\frac{\partial\theta}{\partial t} + u_k \frac{\partial \theta}{\partial x_k} = \nabla^2\theta + \mathscr{R}u_3, \tag{2.12}$$

$$\frac{\partial u_i}{\partial x_i} = 0. \tag{2.13}$$

Here $\mathscr{P}$ is the Prandtl number, $\mathscr{R}$ the Rayleigh number and Γ a number proportional to $d\nu/dT$:

$$\mathscr{P} = \frac{\nu_0}{\kappa}, \quad \mathscr{R} = \frac{\alpha g\beta h^4}{\nu_0\kappa}, \quad \Gamma = \frac{\nu_0^2\gamma}{\alpha g h^3} = h\beta\gamma\mathscr{P}/\mathscr{R}. \tag{2.14}$$

The horizontal boundaries may be either rigid or so-called 'free'. In the first case $u_i = 0$ at the boundaries; in the last case the vertical velocity and the shearing stresses are zero at the boundary. It will also be assumed that the temperatures at the horizontal boundaries are maintained constant. Applying (2.13) we thus have

$$u_i = \theta = 0 \quad \text{at rigid boundaries,}$$
$$u_3 = \partial^2 u_3/\partial x_3^2 = \theta = 0 \quad \text{at free boundaries.}$$

3. Application of the perturbation method

It is convenient to introduce a four-dimensional vector

$$v_k = \begin{Bmatrix} \theta \\ u_i \end{Bmatrix},$$

the four-dimensional operator

$$\frac{\partial}{\partial x_k} = \begin{Bmatrix} 0 \\ \dfrac{\partial}{\partial x_i} \end{Bmatrix},$$

and the matrix differential operator

$$D_{ik} = \begin{Bmatrix} \nabla^2 & \mathscr{R}\delta_{3k} \\ \mathscr{P}\delta_{i3} & \mathscr{P}\nabla^2\delta_{ik} \end{Bmatrix}.$$

Furthermore,

$$v_{ik} = \begin{Bmatrix} 0 & 0 \\ 0 & u_{ik} \end{Bmatrix}.$$

Equations (2.11)–(2.13) may then be written

$$\frac{\partial v_i}{\partial t}+v_k\frac{\partial v_i}{\partial x_k} = -\frac{\partial p}{\partial x_i}+D_{ik}v_k-\Gamma\mathscr{R}\frac{\partial(x_3 v_{ik})}{\partial x_k}+\Gamma\frac{\partial(v_0 v_{ik})}{\partial x_k}, \tag{3.1}$$

$$\frac{\partial v_i}{\partial x_i} = 0. \tag{3.2}$$

Intending to apply the perturbation method, we write

$$\mathscr{R} = \mathscr{R}^{(0)}+\epsilon\mathscr{R}^{(1)}+\epsilon^2\mathscr{R}^{(2)}+\ldots, \tag{3.3}$$

$$v_k = \epsilon v_k^{(1)}+\epsilon^2 v_k^{(2)}+\ldots, \tag{3.4}$$

$$p = \epsilon p^{(1)}+\epsilon^2 p^{(2)}+\ldots, \tag{3.5}$$

where $\mathscr{R}$ is considered as known. Introducing (3.3)–(3.5) in (3.1), and utilizing the fact that (3.1) must be satisfied for all values of ϵ, we obtain an infinite set of equations. It is appropriate in these equations to apply the operator

$$L_{ik} = D_{ik}^0-\Gamma\mathscr{R}^{(0)}D_{ik}^\gamma, \tag{3.6}$$

where the superscript on D_{ik}^0 means that $\mathscr{R}$ is replaced by $\mathscr{R}^{(0)}$ and

$$D_{ik}^\gamma = \begin{Bmatrix} 0 & 0 \\ 0 & \delta_{ik}\left(x_3\nabla^2+\dfrac{\partial}{\partial x_3}\right)+\delta_{k3}\dfrac{\partial}{\partial x_i} \end{Bmatrix}. \tag{3.7}$$

It is readily shown that with the actual boundary conditions the operator L_{ik} is self-adjoint. It has been proved by Schlüter *et al.* that D_{ik}^0 is self-adjoint, and it therefore here suffices to prove that also D_{ik}^γ is self-adjoint. We define the weighted scalar product of the two vectors v_k' and v_k'' by

$$\langle v_k', v_k''\rangle = \mathscr{P}\overline{\theta'\theta''}+\mathscr{R}^{(0)}\overline{u_i' u_i''}, \tag{3.8}$$

where the bar denotes the average over the entire layer. Assuming that the vectors are solenoidal vectors and applying the boundary conditions, we then have

$$\langle v_i', D_{ik}^\gamma v_k''\rangle = \mathscr{R}^{(0)}\overline{u_i'\frac{\partial(x_3 u_{ik}'')}{\partial x_k}}$$

$$= -\tfrac{1}{2}\mathscr{R}^{(0)}\overline{x_3 u_{ik}' u_{ik}''} = \langle v_i'', D_{ik}^\gamma v_k'\rangle \tag{3.9}$$

which proves the self-adjointness of the operator.

Taking into account only terms up to the third order, we obtain from (3.1)

$$L_{ik}v_k^{(1)}-\frac{\partial p^{(1)}}{\partial x_i} = 0, \tag{3.10}$$

$$L_{ik}v_k^{(2)}-\frac{\partial p^{(2)}}{\partial x_i} = v_k^{(1)}\frac{\partial v_i^{(1)}}{\partial x_k}+\Gamma\mathscr{R}^{(1)}\frac{\partial(x_3 v_{ik}^{(1)})}{\partial x_k}-\Gamma\frac{\partial(v_0^{(1)}v_{ik}^{(1)})}{\partial x_k}+\frac{\partial v_i^{(1)}}{\partial t}-\mathscr{R}^{(1)}v_3^{(1)}\delta_{i0}, \tag{3.11}$$

$$\begin{aligned} L_{ik}v_k^{(3)}-\frac{\partial p^{(3)}}{\partial x_i} = {} & v_k^{(2)}\frac{\partial v_i^{(1)}}{\partial x_k}+v_k^{(1)}\frac{\partial v_i^{(2)}}{\partial x_k}+\Gamma\mathscr{R}^{(1)}\frac{\partial(x_3 v_{ik}^{(2)})}{\partial x_k}+\Gamma\mathscr{R}^{(2)}\frac{\partial(x_3 v_{ik}^{(1)})}{\partial x_k} \\ & -\Gamma\frac{\partial(v_0^{(2)}v_{ik}^{(1)})}{\partial x_k}-\Gamma\frac{\partial(v_0^{(1)}v_{ik}^{(2)})}{\partial x_k}-\mathscr{R}^{(2)}v_3^{(1)}\delta_{i0}-\mathscr{R}^{(1)}v_3^{(2)}\delta_{i0}+\frac{\partial v_i^{(2)}}{\partial t}. \end{aligned} \tag{3.12}$$

Equation (3.10) is the linearized equation determining the critical Rayleigh number $\mathscr{R}^{(0)}$. Due to the self-adjointness of L_{ik}, $\partial/\partial t$ does not enter in this equation. The necessary and sufficient condition to secure that (3.11) and (3.12) are solvable, is that the vector on the right-hand side is orthogonal, in the sense of (3.8), to all the solutions of (3.10). From equation (3.11) we then obtain

$$\left\langle v_i^{(1)'}, v_k^{(1)}\frac{\partial v_i^{(1)}}{\partial x_k}\right\rangle + \Gamma\mathscr{R}^{(1)}\left\langle v_i^{(1)'}, \frac{\partial(x_3 v_{ik}^{(1)})}{\partial x_k}\right\rangle - \Gamma\left\langle v_i^{(1)'}, \frac{\partial(v_0^{(1)}v_{ik}^{(1)})}{\partial x_k}\right\rangle$$
$$+\left\langle v_i^{(1)'}, \frac{\partial v_i^{(1)}}{\partial t}\right\rangle - \mathscr{R}^{(1)}\langle v_i^{(1)'}, v_3^{(1)}\rangle\,\delta_{i0} = 0, \quad (3.13)$$

where $v_i^{(1)'}$ denotes an arbitrary solution of (3.10). According to Schlüter *et al.*

$$\left\langle v_i^{(1)'}, v_k^{(1)}\frac{\partial v_i^{(1)}}{\partial x_k}\right\rangle = 0. \quad (3.14)$$

From equation (2.14) it follows that $\Gamma\mathscr{R}^{(0)} = \gamma h\beta\mathscr{P} = \mathscr{P}\Delta\nu/\nu_0$, where $\Delta\nu$ is the difference in viscosity between the two horizontal boundaries. (3.13) may then after some simple manipulation be written

$$\mathscr{P}\mathscr{R}^{(1)}\left(\overline{\theta^{(1)'}u_3^{(1)}} + \frac{\Delta\nu}{\nu_0}\overline{x_3\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i}{\partial x_k}}\right) = \left\langle v_i^{(1)'}, \frac{\partial v_i^{(1)}}{\partial t}\right\rangle - \Gamma\mathscr{R}^{(0)}\overline{u_i^{(1)'}\frac{\partial(\theta^{(1)}u_{ik}^{(1)})}{\partial x_k}}. \quad (3.15)$$

We shall assume that $\Delta\nu/\nu_0$ is a relatively small quantity so that the last term in the parentheses may be neglected. This is a very good approximation also for moderate values of $\Delta\nu/\nu_0$. Indeed if the origin of the frame of references is placed in the middle of the fluid layer, this term is exactly zero for symmetrical boundary conditions (the free-free and the rigid-rigid case). (3.15) therefore reduces to

$$\mathscr{P}\mathscr{R}^{(1)}\overline{\theta^{(1)'}u_3^{(1)}} = \left\langle v_i^{(1)'}, \frac{\partial v_i^{(1)}}{\partial t}\right\rangle - \Gamma\mathscr{R}^{(0)}\overline{u_i^{(1)'}\frac{\partial(\theta^{(1)}u_{ik}^{(1)})}{\partial x_k}}, \quad (3.16)$$

which determines $\mathscr{R}^{(1)}$.

Correspondingly we obtain from (3.12) an equation which determines $\mathscr{R}^{(2)}$. According to Schlüter *et al.*

$$\left\langle v_i^{(1)'}, v_k^{(2)}\frac{\partial v_i^{(1)}}{\partial x_k}\right\rangle = 0.$$

Assuming $\Delta\nu/\nu_0$ relatively small, we then obtain

$$\mathscr{P}\mathscr{R}^{(2)}\overline{\theta^{(1)'}u_3^{(1)}} = \left\langle v_i^{(1)'}, \frac{\partial v_i^{(2)}}{\partial t}\right\rangle - \mathscr{P}\mathscr{R}^{(1)}\overline{\theta^{(1)'}u_3^{(2)}} + \left\langle v_i^{(1)'}, v_k^{(1)}\frac{\partial v_i^{(2)}}{\partial x_k}\right\rangle. \quad (3.17)$$

In our derivation the amplitude of the motion is not normalized. We may therefore in (3.3) choose $\epsilon = 1$, obtaining the same equation as would have been derived by using the straightforward iteration process. Thus

$$\mathscr{R} - \mathscr{R}^{(0)} = \Delta\mathscr{R} = \mathscr{R}^{(1)} + \mathscr{R}^{(2)}, \quad (3.18)$$

where $\Delta\mathscr{R}$ is considered as a known quantity.

4. Derivation of the non-linear amplitude equations

As mentioned in the introduction we shall consider a motion which to the first order consists of only two Fourier components such that the vertical velocity takes the form

$$u_3^{(1)} = A_{11}(t)f(z)\cos kx \cos ly + A_{02}(t)f(z)\cos 2ly, \tag{4.1}$$

where

$$k^2+l^2 = 4l^2 = a^2 \tag{4.2}$$

and x, y, z are used instead of x_1, x_2, x_3, respectively. The overall wave number a will be set equal to a_c, its value at the onset of convection. For $\Gamma = 0$, the function $f(z)$ may (Pellew & Southwell 1940; Reid & Harris 1958) be written:

rigid-rigid case

$$f(z) = \sum_{n=1}^{3} a_n \cosh a\mu_n z, \quad -\tfrac{1}{2} \leqslant z \leqslant \tfrac{1}{2}, \quad a_c = 3{\cdot}117,$$
$$\mu_1 = 1{\cdot}275i, \quad \mu_2 = 1{\cdot}667\text{–}0{\cdot}682i, \quad \mu_3 = \mu_2^*,$$
$$a_1 = 1, \quad a_2 = -0{\cdot}03076+0{\cdot}05194i, \quad a_3 = a_2^*;$$

free-free case

$$f(z) = \sum_{n=1}^{3} a_n \cosh a\mu_n z, \quad -\tfrac{1}{2} \leqslant z \leqslant \tfrac{1}{2}, \quad a_c = \pi/\sqrt{2},$$
$$\mu_1 = \sqrt{2}\,i, \quad \mu_2 = \mu_3 = 0,$$
$$a_1 = 1, \quad a_2 = a_3 = 0;$$

rigid-free case

$$f(z) = \sum_{n=1}^{3} a_n \sinh a\mu_n z, \quad -1 \leqslant z \leqslant 0, \quad a_c = 2{\cdot}682,$$
$$\mu_1 = 1{\cdot}331i, \quad \mu_2 = 1{\cdot}698-0{\cdot}706i, \quad \mu_3 = \mu^*,$$
$$a_1 = -i, \quad a_2 = -0{\cdot}00854+0{\cdot}00173i, \quad a_3 = a_2^*.$$

The values of $u_1^{(1)}$ and $u_2^{(1)}$ corresponding to (4.1) are easily found from the continuity equation and vorticity equation, and $\theta^{(1)}$ is found by elimination of p in (2.11) and application of the continuity equation.

The second-order terms are found from (3.11). Since $\Delta\nu/\nu_0$ is assumed relatively small, all terms on the right-hand side except the first, may be cancelled. In this term Γ may be set equal to zero such that $f(z)$ in (4.1) takes the values given above. By elimination of the pressure we then obtain

$$(\nabla^6 - \mathscr{R}^{(0)}\nabla_2^2)\, u_3^{(2)} = -\mathscr{P}^{-1}\nabla^2\delta_i\, u_j^{(1)}\frac{\partial u_i^{(1}}{\partial x_j} - \nabla_2^2\, u_j^{(1)}\frac{\partial \theta^{(1)}}{\partial x_j}, \tag{4.3}$$

where ∇_2^2 is the two-dimensional Laplace operator and

$$\delta_i = \frac{\partial^2}{\partial x_i\, \partial x_3} - \delta_{i3}\nabla^2.$$

The solution of (4.3) is after some calculations found to be given by

$$u_3^{(2)} = \sum_{i,j} K_{ij}F_n(z)\cos ikx \cos jly, \tag{4.4}$$

where (i,j) may take the six values

$$(i,j) = (1,1),\ (0,2),\ (1,3),\ (2,0),\ (0,4), (2,2). \tag{4.5}$$

$F_n(z)$ and K_{ij} are defined in the appendix.

$u_1^{(2)}$, $u_2^{(2)}$ and $\theta^{(2)}$ are found in a similar way as the corresponding first-order terms (for details see, for example, Palm (1960)). $\theta^{(2)}$ takes the form

$$\theta^{(2)} = a^2 \sum_{i,j} K_{ij} G_n(z) \cos ikx \cos jly, \tag{4.6}$$

where (i,j) in addition to the values (4.5) also takes the value $(0,0)$. K_{ij} and $G_n(z)$ are defined in the appendix.

Introducing our expressions for the first- and second-order terms in (3.18), we find after some calculations the amplitude equations

$$K\dot{A}_{11} = EA_{11} - AA_{11}A_{02} - RA_{11}^3 - PA_{11}A_{02}^2, \tag{4.7}$$

$$K\dot{A}_{02} = EA_{02} - \tfrac{1}{4}AA_{11}^2 - R_1 A_{02}^3 - \tfrac{1}{2}PA_{11}^2 A_{02}, \tag{4.8}$$

which are valid to the third order in the amplitude and for small values of $\Delta\nu/\nu_0$. Here

$$K = \mathscr{R}^{(0)} \int fDf\,dz + a^2\mathscr{P} \int (D^2f)^2\,dz, \tag{4.9}$$

$$E = \mathscr{P}\Delta\mathscr{R}a^2 \int fD^2f\,dz, \tag{4.10}$$

$$A = \tfrac{1}{2}\Gamma\mathscr{R}^{(0)} \int D^2f\{5a^2f'^2 + (f'' + a^2f)^2\}\,dz, \tag{4.11}$$

$$\begin{aligned} R = -\tfrac{1}{64}\mathscr{R}^{(0)} &\int \{F_1(fDf' + 2f'Df) + 3F_3\,fDf' + 4F_4(fDf' - f'Df)\}\,dz \\ &- \frac{a^3}{64}\mathscr{P} \int \{8G_0(fD^2f' + f'Df) + G_1(2fD^2f' + f'D^2f) \\ &\qquad + G_3(2fDf' - f'D^2f) + 2G_4(fD^2f' - f'D^2f)\}\,dz, \end{aligned} \tag{4.12}$$

$$\begin{aligned} R_1 = -\frac{1}{4a}\mathscr{R}^{(0)} &\int F_4(fDf' - f'Df)\,dz \\ &- \frac{a^3}{8}\mathscr{P} \int \{2G_0(fD^2f' + f'D^2f) + G_4(fD^2f' - f'D^2f)\}\,dz, \end{aligned} \tag{4.13}$$

where
$$D = 1 - a^{-2}\frac{\partial^2}{\partial z^2}$$

and
$$P = 4R - R_1. \tag{4.14}$$

The integral sign denotes integration over the fluid depth, i.e. $-\tfrac{1}{2} \leqslant z \leqslant \tfrac{1}{2}$ for the rigid-rigid case and the free-free case, and $-1 \leqslant z \leqslant 0$ for the rigid-free case.

5. Discussion of the amplitude equations

Equations (4.7) and (4.8) yield the time development of the amplitudes A_{11} and A_{02}, and are formally the same equations as discussed in I–IV. Let

$$\chi = \frac{\Delta\mathscr{R}}{\mathscr{R}^{(0)}} \bigg/ \left(\frac{\Delta\nu}{\nu_0}\right)^2, \tag{5.1}$$

$$A = \Gamma\mathscr{R}^{(0)}A' \tag{5.2}$$

and

$$E = \mathscr{P}\Delta\mathscr{R}E'. \tag{5.3}$$

We then have that the transition between the regimes of hexagons and hexagons + rolls takes place for $\chi = \chi_1$, where

$$\chi_1 = \frac{R_1}{4(2R-R_1)^2}\frac{\mathscr{P}}{\mathscr{R}^{(0)}}\frac{A'^2}{E'} \tag{5.4}$$

and the transition between the regimes of hexagons + rolls and rolls takes place for $\chi = \chi_2$ where

$$\chi_2 = \frac{4R+R_1}{R_1}\chi_1. \tag{5.5}$$

Evaluation of the integrals in (4.9)–(4.13) leads to:

rigid-rigid case

$E' = 30{\cdot}77$,	$R = 200{\cdot}0\mathscr{P} + 10{\cdot}57\mathscr{P}^{-1} + 10{\cdot}43$,
$A' = 577{\cdot}6$,	$R_1 = 330{\cdot}8\mathscr{P} + 2{\cdot}402\mathscr{P}^{-1} - 0{\cdot}836$,
$\mathscr{P} = 0{\cdot}7$ (air),	$\chi_1 = 0{\cdot}028$, $\chi_2 = 0{\cdot}106$,
$\mathscr{P} = 6{\cdot}0$ (water),	$\chi_1 = 0{\cdot}098$, $\chi_2 = 0{\cdot}336$,
$\mathscr{P} = 3500$ (Silicone oil AK 350),	$\chi_1 = 0{\cdot}109$, $\chi_2 = 0{\cdot}385$,

free-free case

$E' = 22{\cdot}21$,	$R = 79{\cdot}14\mathscr{P} + 6{\cdot}555\mathscr{P}^{-1} + 4{\cdot}222$,
$A' = 279{\cdot}1$,	$R_1 = 123{\cdot}3\mathscr{P}$,
$\mathscr{P} = 0{\cdot}7$ (air),	$\chi_1 = 0{\cdot}030$, $\chi_2 = 0{\cdot}127$,
$\mathscr{P} = 6{\cdot}0$ (water),	$\chi_1 = 0{\cdot}122$, $\chi_2 = 0{\cdot}437$,
$\mathscr{P} = 3500$ (Silicone oil AK 350),	$\chi_1 = 0{\cdot}133$, $\chi_2 = 0{\cdot}483$,

rigid-free case

$E' = 26{\cdot}47$,	$R = 132{\cdot}2\mathscr{P} + 6{\cdot}430\mathscr{P}^{-1} + 8{\cdot}645$,
$A' = 409{\cdot}3$,	$R_1 = 213{\cdot}1\mathscr{P} - 1{\cdot}625\mathscr{P}^{-1} + 1{\cdot}975$,
$\mathscr{P} = 0{\cdot}7$ (air),	$\chi_1 = 0{\cdot}029$, $\chi_2 = 0{\cdot}115$,
$\mathscr{P} = 6{\cdot}0$ (water),	$\chi_1 = 0{\cdot}104$, $\chi_2 = 0{\cdot}367$,
$\mathscr{P} = 3500$ (Silicone oil AK 350),	$\chi_1 = 0{\cdot}116$, $\chi_2 = 0{\cdot}405$.

The results above for the free-free case may be compared with those obtained in Segel (1965) (where a slightly different law of viscosity variation is used) and the agreement is very good. The values for R and R_1 in the rigid-rigid case may be compared with those obtained by Schütler *et al.* and the agreement is good for the leading terms. For the small terms there are, however, some discrepancies. We have not been able to find the reason for this. For the Prandtl numbers in question however, these terms are so small that they do not influence the result.

6. Conclusion

Both hexagons and rolls may be observed for Rayleigh numbers for which the corresponding χ-values lie between χ_1 and χ_2. Which of these modes shall be realized depends on the initial conditions. Thus, if the Rayleigh number is increased slowly, hexagons will be observed up to the value χ_2. Above this value the motion will consist of rolls. If the Rayleigh number now is decreased slowly, rolls will occur down to the value χ_1. Below this value, a hexagonal pattern will be observed.

It is seen from the values above that χ_1 and χ_2 are nearly the same for the rigid-rigid case, the free-free case and the rigid-free case. The Rayleigh numbers corresponding to the transitions between the various modes are given by

$$\Delta\mathscr{R}/\mathscr{R}^{(0)} = \chi_{1,2}\left(\frac{\Delta\nu}{\nu_0}\right)^2 = \chi_{1,2}\left(\frac{\kappa\nu_0\gamma}{\alpha g h^3}\right)^2 \mathscr{R}^{(0)2}, \tag{6.1}$$

where $\chi_{1,2}$ is χ_1 or χ_2. The parentheses on the right contain only fluid properties, the acceleration of gravity and the depth of the fluid layer. Therefore, in experiments with the same fluid and fluid depth, but with different boundary conditions, $\Delta\mathscr{R}/\mathscr{R}^{(0)}$ will in the three cases considered be approximately proportional to the square of the critical Rayleigh number. Thus the regime of cellular motion will take place for a larger interval of the Rayleigh number in the rigid-rigid case than in the two other cases.

The most outstanding feature of (6.1) is, however, that $\Delta\mathscr{R}/\mathscr{R}^{(0)}$ is proportional to the inverse of *the fluid depth to the sixth power*. The extent of the interval of the Rayleigh number for which hexagons are to be observed is therefore extremely sensitive to the choice of the fluid depth in the experiments. As an example, let us consider Silveston's (1958) experiments on thermal convection with rigid-rigid boundaries. From his data it follows that in the case of a fluid layer of a depth 7 mm and an average temperature of 30 °C, at the onset of convection $\Delta\nu/\nu_0$ is about 0·43 for Silicone oil AK 350. From the formulas above it then follows that the transition from the regime of hexagons to the regime of hexagons + rolls takes place for $\Delta\mathscr{R}/\mathscr{R}^{(0)} = 2\,\%$ and the transition from the regime of hexagons + rolls to the regime of rolls for $\Delta\mathscr{R}/\mathscr{R}^{(0)} = 7\,\%$. If, on the other hand, the depth of the same fluid had been chosen as 5 mm, the corresponding transition values must be multiplied with a factor $(\frac{7}{5})^6 = 7{\cdot}53$ giving $\Delta\mathscr{R}/\mathscr{R}^{(0)} = 15\,\%$ and $\Delta\mathscr{R}/\mathscr{R}^{(0)} = 54\,\%$, respectively.

Since appropriate experimental investigations of the transition values are not available, a direct comparison between theory and experiments is not possible. It may, however, be mentioned that Koschmieder (1966) in his experiments did not observe hexagons in the rigid-rigid case. It turns out that he used a 10 mm deep layer of Silicone oil and only examined the pattern for $\Delta\mathscr{R}/\mathscr{R}^{(0)} = 20\,\%$. According to the theory, hexagons are not stable for this value of the Rayleigh number. However, with rigid-free boundaries and a depth of 4 mm, he observed a very stable hexagonal pattern. According to the formulas above,† the transition Rayleigh number for hexagons/hexagons + rolls is given by $\Delta\mathscr{R}/\mathscr{R}^{(0)} = 25\,\%$ and the transition Rayleigh number for hexagons + rolls/rolls by $\Delta\mathscr{R}/\mathscr{R}^{(0)} = 86\,\%$.

† Strictly speaking, the 'free' boundary conditions as applied here are not the correct boundary conditions for a free surface, the elevation of this being neglected. Observations show, however, that this elevation is very small. Furthermore, with the correct free boundary conditions the principle of exchange of stabilities is probably not strictly true, since gravity surface waves may be set up. It does not, however, seem reasonable that this effect is of importance.

It must be pointed out that in the derivation of the formulas above, we assumed that the amplitude of the motion is small, and that $\Delta\nu/\nu_0$ is a relatively small quantity. We believe that our approximations are fairly good in the case with a fluid depth 7 mm considered above. For fluid depth of 4 mm, we surely stretch our formulas too far. However, the qualitative result that the interval of Rayleigh numbers where hexagons occur, increases strongly when the fluid depth is lowered, remains correct. It is pertinent in this connexion to point out that the occurrence of a hexagonal pattern is, to our knowledge, always strongly related to thin layers. As is well known, Bénard (1901) in his experiments applied fluid depths down to 1 mm. For thin layers hexagons should be observed as well in the rigid-rigid case as in the rigid-free case. In the first case there will be no disturbing effects due to the surface tension.

As in the case of free-free boundaries, the critical Rayleigh number will be lowered somewhat due to the variation of ν with temperature. This effect is not computed here. For the free-free case we refer to Palm (1960) and for the rigid-rigid case to Jenssen (1963). To obtain a similar formula also for the rigid-free case, ν_0 must be interpreted as the value of ν in a certain point near the middle of the fluid layer. As to the occurrence of subcritical Rayleigh numbers due to non-linear effects, we refer to Segel & Stuart (1962) and Segel (1965). It should also be pointed out that, as shown in Palm (1960) and Segel & Stuart (1962), the hexagons are characterized by ascent or descent in the middle of the cell according as the viscosity decreases or increases with temperature.

We have here only taken into account the variation of ν with temperature. Also other material properties may vary. These variations are, however, usually much smaller than the variation of ν. Also the effect of surface tension has been neglected in this paper.

Appendix

The second-order terms $u_3^{(2)}$ and $\theta^{(2)}$ have the forms (4.4) and (4.6). The index n is defined by $(ik)^2+(jl)^2 = na^2$; n thus takes the values 1, 3, 4 and 0. The amplitude factors K_{ij} are defined by

$$K_{11} = A_{11}A_{02}/4a, \qquad K_{02} = A_{11}^2/16a,$$
$$K_{13} = 3A_{11}A_{02}/4a, \qquad K_{20} = 3A_{11}^2/16a,$$
$$K_{04} = A_{02}^2/a, \qquad K_{22} = A_{11}^2/2a,$$
$$K_{00} = (A_{11}^2+2A_{02}^2)/8a.$$

For the functions $F_n(z)$ the following expressions are obtained for $n = 1, 3, 4$:

$$F_n(z) = \sum_{\alpha,\beta=1}^{3} a_\alpha a_\beta H_n\left(\frac{\sinh(\mu_\alpha+\mu_\beta)\,az}{\Delta_n^+} \pm \frac{\sinh(\mu_\alpha-\mu_\beta)\,az}{\Delta_n^-}\right) + \sum_{\alpha=1}^{3} b_{n\alpha}\sinh q_{n\alpha}az,$$

where we have set

$$H_1 = \mu_\beta(2\lambda_\beta^2+\lambda_\alpha^2)+\mathscr{P}^{-1}\mu_\beta\{\lambda_\beta(\lambda_\beta+7\lambda_\alpha-5)+\lambda_\alpha(4\lambda_\alpha-1)\},$$
$$H_3 = \mu_\beta(2\lambda_\beta^2-\lambda_\alpha^2)+\mathscr{P}^{-1}\mu_\beta\{\lambda_\beta(\lambda_\beta+\lambda_\alpha+1)+2\lambda_\alpha(\lambda_\alpha-1)\},$$
$$H_4 = (\lambda\,\mu_{\beta\beta}^2-\lambda_\alpha^2)+\tfrac{1}{2}\mathscr{P}^{-1}\mu_\beta\{\lambda_\beta(\lambda_\beta-2\lambda_\alpha+4)+\lambda_\alpha(\lambda_\alpha-4)\},$$

and
$$\lambda_\alpha = 1-\mu_\alpha^2, \quad q_{n\alpha} = n+n^{\frac{1}{3}}(\mu_\alpha^2-1),$$
$$\Delta_n^{\pm} = \{(\mu_\alpha \pm \mu_\beta)^2 - n\}^3 + n\mathscr{R}^{(0)}/a^4.$$

$b_{n\alpha}$ are constants which have to be determined such that $u_3^{(2)}$ satisfies the proper boundary conditions. For the rigid-free case the upper sign has to be used, while the lower sign applies to the rigid-rigid and the free-free cases.

The corresponding expressions for the $G_n(z)$ for $n = 1, 3$ and 4 are:

$$G_n(z) = \sum_{\alpha,\beta=1}^{3} a_\alpha a_\beta H_n \left(\frac{\sinh(\mu_\alpha+\mu_\beta)az}{D_n^+} \pm \frac{\sinh(\mu_\alpha-\mu_\beta)az}{D_n^-}\right)$$
$$+\mathscr{P}^{-1}\sum_{\alpha,\beta=1}^{3} a_\alpha a_\beta H_n'\{\sinh(\mu_\alpha+\mu_\beta)az \pm \sinh(\mu_\alpha-\mu_\beta)az\}$$
$$+\sum_{\alpha=1}^{3} b_{n\alpha}(q_{n\alpha}^2-n)^2 \sinh q_{n\alpha} az.$$

In addition to the quantities defined above we have introduced
$$D_n^{\pm} = \{(\mu_\alpha \pm \mu_\beta)^2 - n\}^{-2}\Delta_n^{\pm},$$
$$H_1' = \mu_\beta(\lambda_\beta + 2\lambda_\alpha), \quad H_3' = \mu_\beta \lambda_\beta, \quad H_4' = \tfrac{1}{2}\mu_\beta(\lambda_\beta - \lambda_\alpha).$$

Finally we quote the function $G_0(z)$
$$G_0(z) = \sum_{\alpha,\beta=1}^{3} a_\alpha a_\beta(1-\mu_\alpha^2)^2\left(\frac{\sinh(\mu_\alpha+\mu_\beta)az}{\mu_\alpha+\mu_\beta} \mp \frac{\sinh(\mu_\alpha-\mu_\beta)az}{\mu_\alpha-\mu_\beta} - Cz\right),$$
where C is a constant which may be written
$$C = \frac{\sinh(\mu_\alpha+\mu_\beta)a}{\mu_\alpha+\mu_\beta} - \frac{\sinh(\mu_\alpha-\mu_\beta)a}{\mu_\alpha-\mu_\beta}$$
for the rigid-free case, and
$$C = 2\left(\frac{\sinh(\mu_\alpha+\mu_\beta)a/2}{\mu_\alpha+\mu_\beta} + \frac{\sinh(\mu_\alpha-\mu_\beta)a/2}{\mu_\alpha-\mu_\beta}\right)$$
for the rigid-rigid and the free-free cases.

REFERENCES

Bénard, H. 1901 *Ann. Chem. Phys.* **23**, 62.
Jenssen, O. 1963 *Acta Polytechnica Scandinavica Ph.* 24.
Koschmieder, E. L. 1966 *Beitr. Phys. Atmos.* **39**, 1.
Malkus, W. V. R. & Veronis, G. 1958 *J. Fluid Mech.* **4**, 225.
Palm, E. 1960 *J. Fluid Mech.* **8**, 183.
Palm, E. & Øiann, H. 1964 *J. Fluid Mech.* **19**, 353.
Pellew, A. & Southwell, R. V. 1940 *Proc. Roy. Soc.* A **176**, 312.
Reid, W. H. & Harris, D. L. 1958 *Phys. Fluids*, **1**, 102.
Schlüter, A., Lortz, D. & Busse, F. 1965 *J. Fluid Mech.* **23**, 129.
Segel, L. A. 1965 *J. Fluid Mech.* **21**, 359.
Segel, L. A. & Stuart, J. T. 1962 *J. Fluid Mech.* **13**, 286.
Silveston, P. L. 1958 *Forsch. Ing.* **24**, 29–32 and 59–69.